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Optimizing Infrastructure Resilience under Budgetary Constraint

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Abstract

Communities located in regions prone to natural and man-made disasters endure hardship and financial loss in the face of these events. Investment to enhance infrastructure resilience is vital to reduce the consequences of these low probability high-impact events. Budget and resources are limited, and they must be allocated wisely to infrastructure components to build a resilient community. The complexity of infrastructure makes it difficult to show the effects of component enhancements on system resilience. This paper proposes a mathematical programming model aimed at optimizing infrastructure resilience against a set of adverse events by optimally allocating budget to the infrastructure components. Investment, component enhancement, and corresponding functionality are combined with the resilience-based component importance to tackle the system complexity. Three utility functions are presented to determine the possible component enhancement alternatives for an allocated budget and to choose the optimal alternative. A resilience-based component importance metric is introduced, which is used in the budget allocation optimization problem. This approach establishes a relationship between amount allocated to a component and changes in its absorption and recovery, and the aggregate of all such changes on the components on the system functionality. The results show that the utility function of a component impacts the resilience enhancement of the system.

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1. Introduction

Adverse events like natural disasters (e.g., earthquake, tropical cyclone, severe storm, flooding, freeze, wildfire, winter storm, etc.) or man-made disasters (terrorist and non-terrorist) can disrupt the community infrastructures. These adverse events have two traits in common. First, their occurrence probability is low; the expected number of hurricanes in 100 years in Texas is 7.1 and the expected number of major hurricanes is 2.2 [1]. Second, their impact in terms of cost and hardship is tremendous. During the past 37 years, 40 cyclones have caused a combined $870.2 billion in total damages with an average of $21.8 billion per event. Hurricane Harvey in 2017 alone accounts for $125 billion of this amount [2]. Resilience is a concept that addresses the ability of a system to continue its functionality during and after an extreme event with low functionality degradation and a rapid return to normalcy [3]. Anticipation, absorption, adaptation, and rapid recovery are the main characteristics of a resilient system [4], [5]. The United States Government Accountability Office (GAO) expressed the necessity of an investment strategy for resilience enhancement that reduces the nation's losses from future disasters. The investment decisions on cyber-physical systems (CPSs [6]), especially those with a social impact like critical infrastructures, can be viewed from political [7], social, environmental [8], economic [9], financial [10], and engineering [7], [11] perspectives. While the benefits of these investments are generally difficult to monetize [12], an early investment in community resilience will pay back when disasters inevitably strike [13], and a lack of investment will possibly result in an overall higher cost [14]. Considering this importance, the Federal Emergency Management Agency (FEMA) is working on the National Mitigation Investment Strategy ("Investment Strategy") which provides a national approach to invest in mitigation activities and risk management across the United States [15].

The problem of investing in urban infrastructure resilience can be considered at three levels: macro, meso, and micro. At the macro level, the problem can be categorized as funding, prioritizing, and resource allocation to several infrastructures. Some of the funding approaches are earmarks [16], pork-barrel [17], trust-fund [18], and block grants [19]. For resource allocation at a macro level, Hill *et al.* [20] suggested a method to reduce disaster impacts through systematic investments in which the socioeconomic risks associated with natural disasters is estimated. Graeden et. al. [21] proposed a rapid risk analysis that can be utilized to support risk-based investment prioritization at the community level. After the budget is assigned to a system, at the meso level, the system allocates resource to its components. For this purpose, one approach is to find the most critical components in the network and improve them. A resilience-based component importance (RCI) metric, which measures the extent to which individual component contributes to the network resilience [22], can be used for this purpose [23]–[25]. Component enhancement can result in a combination of less degradation in component functionality and rapid recovery of functionality in the face of a shock. At the micro level, the problem considers this combination.

Literature has covered some specific issues and events regarding this subject, but there is more that needs to be taken into account. Some studies considered just a special system (transportation [26], [27], power grid [12], [28], etc.), or a single event (e.g., cyber-attack [29], terrorist attacks [28], etc.), and they suggested a treatment for that specific system or event. Even so, a system is threatened by a pool of events, and preparing for only one of them and neglecting the other can still be devastating if a second event strikes while attempting to recover from the first one. Moreover, if we consider the pool of potential events, the amount of investment to mitigate multiple risks is more efficacious than investing against each risk individually. The approaches based on RCI [23]–[25] do not provide the amount of the investment on the components. While RCI ranks the components based on their importance, it does not determine how much should be invested in each one. Moreover, the allotted resource to a component can be utilized in different ways to change the main resilience characteristics of that component, i.e., absorption and time to recovery. RCI does not determine which of these characteristics must be emphasized. Hence, this paper attempts to fill these gaps by proposing a method that can be applied to a general system. Our contributions include a novel formulation, introduction of utility function into component enhancement, and a component importance metric. The proposed method takes into account the set of possible events, their effects on the component functionality, and the component's enhancement alternatives. The alternatives employ a utility function to construct the set of alternatives for component enhancement. We also introduce a resilience-based component importance metric. In the final step of the method, a mathematical programming model is introduced that incorporates the information we generated in the previous steps and optimizes the system resilience of under a budget constraint.

2. Model & Solution Methodology

The resource allocation problem has many applications in facility planning, job scheduling, buffer allocation, pollution control, and portfolio management. The investment on a system resilience can be translated into a resource allocation problem. In this section, we start with a tradeoff between absorption and recovery enhancements for a single component. Then, we propose an RCI followed by an integer programming formulation that optimally allocates budget to the components aiming at maximizing the system resilience.

2.1. Single component functionality and indifference curve

In order to develop a mathematical model and have a better understanding of the system behavior, we study the components individually. The functionality of a component is the level at which the component performs a task or function. For example, the functionality of a water transmission pipeline is the amount of flow that it carries. In a normal situation, the *target functionality* is the amount of water that the pipe is planned and expected to carry. Two component's characteristics that influence its functionality during and after an adverse event are absorbability and rapid recovery. After an event, the functionality degrades by A. Absorbability is the ability to reduce the negative effects of the event and have a smaller A. Rapid recovery or recovery is the time to recover (T) from a disruption. It is the length of the time from the moment that the event happens to the moment that the functionality returns to an acceptable level, usually the initial level. Keeping all other factors fixed, a component with a smaller A or a smaller T has a higher resilience. In this section, we study different outcomes of enhancement activities on a single component. We will use the following notation.

A: The amount of degradation in the functionality
T: Recovery time of the component
a: Improvement in the absorption (in percent)
r: Improvement in the recovery time of the component (in percent)
f: Functionality of the component
Tf: Target functionality of the component
y: Amount of investment on the component
V: Value of the component

The functionality f here is normalized by dividing the real functionality by the target functionality, yielding a value between 0% and 100%. The 100% functionality occurs in the normal situation or for a system that is highly resilient. Different resilience enhancement activities can result in different outcomes for improving absorbability and recovery. Let a and r stand for the percent improvement in the absorption and recovery, respectively. The smallest value of a (or r) is 0, meaning that we do not improve the component's absorbability (recovery time), and the highest value is 1, for which the component will be intact by an event. For a given amount of investment, y, we may have different combinations of (a, r) (Figure 1). If we spend all the money on the robustness of the component, it will improve absorption by a percent (Figure 1-b) and A_{new} will be $A \times (1 - a)$. However, if we spend all the money on redundancy, it may just reduce time to recovery by r (i.e., $T_{new} = T \times (1 - r)$ as in Figure 1-c). Moreover, we may be able to improve both capabilities together (Figure 1-d). In an extreme case, if a is 100%, then there is no degradation in the component functionality and, hence, there is no need to improve the time to recovery and vice versa.

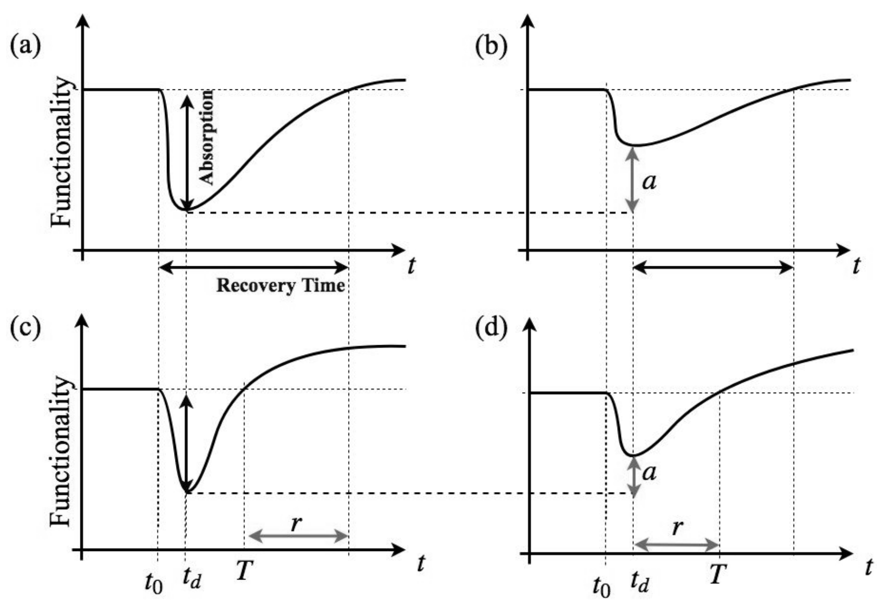

Figure 1: component resilience enhancement scenarios for a fixed budget. System's functionality after disaster and a) before any enhancement, b) after enhancing absorbability, c) after reducing the recovery time, d) after improving both absorbability and recovery time.

The relationship between the investment amount and (a, r) can be established using the indifference curves. In economics, indifference curves represent different quantities of two goods for which a consumer has no preference for one combination of those goods over another combination on the same curve [30]. Using this concept, we define the component enhancement indifference curve (IC) to be all combinations of (a, r) for which we will have the same enhancement cost. It represents different types of improvements that we can perform for a fixed cost. Associated with indifference curves, there is a utility function $U(a, r)$ which relates the budget spent on a component and the improvements in its absorption and recovery time (Equation 1).

$$\begin{aligned} U: [0,1] \times [0,1] &\to R^{+} \\ (a, r) &\to U(a, r) \\ y &= U(a, r) \end{aligned} \tag{1}$$

For example, in Figure 2, the cost of improving absorption and recovery corresponding to points $P1: (a_1, r_1)$ and $P2: (a_2, r_2)$ are $y(P_1) = U(a_1, r_1)$ and $y(P_2) = U(a_2, r_2)$. We assume that the ICs are *complete*, in a way that all points on the indifference curve cost the same amount, and the points not on the curve cost either more or less. Figure 2 shows three investment costs y_1, y_2, and y_3, where $y_1 < y_2 < y_3$. Since points P_1 and P_2 in Figure 2 are on the same indifference curve, they have the same cost of $y(P_1) = y(P_2) = y_1$. Another characteristic of the IC curves is that they have a negative slope. That is, if a is decreased, r should be increased to stay on the same IC. Linear [31], Cobb–Douglas [31], and Constant Elasticity Substitution (CES) [32] are examples of utility curves (Table 1). Value of the component (V) is the market value of the component. Consider two components with values of V_1 and V_2, where V_1 is much larger than V_2. To maintain the same enhancements for both components (i.e., $a_1 = a_2$ and $r_1 = r_2$), the amount of investment on the second component should be larger (i.e., $y_1(P_1) > y_2(P_1)$). It is assumed that y_i is proportionate to the V_i and it is incorporated into the model by scale factor k_i. We call the result of multiplication k by $U(a,r)$ the cost factor. Having the cost factor θ_i and the value V_i for the component i, the amount of investment on component will be

$$y_i = \theta_i V_i.$$

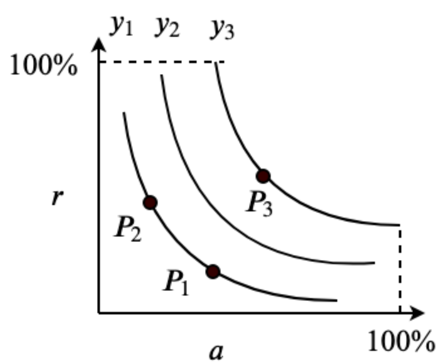

Figure 2 Indifference curve for a single component

Table 1: Utility functions and the relationship of a and r

name	formula	relationships
Linear [31]	$U(a,r) = \frac{\beta_1 a + \beta_2 r}{k}$	$r = \frac{ky - \beta_1 a}{\beta_2}$
Cobb–Douglas [31]	$U(a,r) = \frac{a^{\rho} r^{1-\rho}}{k}, \rho < 1$	$r = e^{\frac{\ln(ky) - \rho \ln(a)}{(1-\rho)}}$
CES [32]	$U(a,r) = \frac{C}{k}(\beta a^{\rho} + (1-\beta) r^{\rho})^{\frac{1}{\rho}}$	$r = \frac{((ky)^{\rho} - \beta a^{\rho})^{\frac{1}{\rho}}}{1-\beta}$

We will use the utility curve to formulate our mathematical model and to determine the optimal combination of a and r on the associated IC curve for a given budget.

2.2. Resilience-Based Component Importance (RCI)

In reliability component importance metrics like Fussell-Vesely, criticality importance measure, risk reduction worth (RRW), risk achievement worth (RAW), and Birnbaum measure the amount by which the failure of a component can affect the reliability of the system [33]. Based on the reliability context, the resilience-based component importance measure (RCI) is defined as the amount by which the resilience of a system is reduced by a component's failure [23], [24]. A prerequisite for calculating the RCI is a resilience metric. Several qualitative and quantitative resilience assessments have been presented [4]. To choose an appropriate metric for a specific analytic, a tiered approach presented in [34] can be used. The metric assessment methodology in [35] will help to select a resilience metric which is a better match for the system under study. Among the quantitative metrics, we need the one that represents the essential characteristics of a resilient system (i.e., absorption and rapid recovery). Moreover, the metric must be simple enough to be utilized in the mathematical formulation and the resulted problem can be solved within a reasonable time. This study uses the resilience metric suggested by Najarian and Lim [35], which has the two mentioned characteristics. It consists of a convex combination of three sub-metrics; absorption ($я_1$), adaptation ($я_2$), and rapid recovery ($я_3$). They can be calculated using the following formulas:

$$я_1 = \frac{\int_{t_o}^{t_d} F(t)dt}{\int_{t_o}^{t_d} TF(t)dt}, я_2 = \frac{\int_{t_d}^{T} F(t)dt}{\int_{t_d}^{T} TF(t)dt}, \text{ and } я_3 = f(T) = \begin{cases} 1, & T \leq T_0 \\ \frac{T_0}{T}, & otherwise \end{cases},$$

where, $F(t)$ is the functionality of the system at time *t*, $TF(t)$ is the target functionality at time *t*, t_d is the time that functionality of the system reaches to its minimum, T_0 is the desired recovery time of the system which , and T is the recovery time of the system. The system resilience metric Я is obtained as in Equation (2) and it lies in the closed interval [0, 1].

$$я = \alpha_1 я_1 + \alpha_2 я_2 + \alpha_3 я_3, \text{ and } \alpha_1 + \alpha_2 + \alpha_3 = 1 \text{ and } \alpha_1, \alpha_2, \alpha_3 \geq 0. \quad (2)$$

Let $я^{+c}$ be the resilience of the system when the component c is operable and $я^{-c}$ when it is not. Then the RCI of the component *i* is the difference between these two values. The following algorithm explains the steps to take to obtain the value of RCI.

Algorithm 1 Resilience-Based Critical Indexing Algorithm

Input the component index $i \in E \cup V$
Calculate the resilience of the system ($я^{+i}$)
Set $f_i = 0$
Calculate the resilience of the system ($я^{-i}$)
$я^i = (я^{+i} - я^{-i})$
Return 1-$я^i$

Since $я^{+i} \geq я^{-i}$, the value of $я^i$ will be within the interval [0,1]; hence, the impact of the component *i* on the system resilience, $RCI_i = (1 - я^i)$, assumes values between 0 and 1, inclusive.

2.3. Resilience Optimization under Budget Constraint

In a complex system, each component has its own absorption and recovery capabilities in the face of an adverse event. For example, in a power grid, even if two components are the same, the environment and the facility that supports them may be different. As a result, their absorptions and recovery times differ, and possible modifications, associated costs, and the effect of enhancement on the whole system must be calculated separately for each component. Consider a power generation unit with possible characteristics to be modified such as elevation, surrounding building and structure, redundancy, source of generation storage (e.g., coal), and environment (e.g., drainages). By improving each or a subset of these characteristics, we can enhance the absorption and/or reduce the recovery time of the generator. This enhancement will differ for different events; a higher elevation or drainage may keep a generator safe against a certain level of the flood, i.e., a better absorption against the flood, but it may not improve it against a hurricane. However, having a backup generator that can immediately become operable in the case of failure in the original generator, shortens the time to recover. Due to the budget limit, a subset of the set of all options should be selected in such a way that a higher resilience level for the system can be achieved. In this section, we are to formulate this problem as and optimization model, and we use the following notation.

Notations:

Sets:

$\Gamma = \{\gamma_1 \dots \gamma_K\}$: Set of k events (including attacks)

$\mathbb{Q} = \{q_1 \dots q_N\}$: Set of N components

$\mathbb{C} = \{c_{i,1} .. c_{i,M_i}\}$: The set of M_i possible investments scenarios on component i

Indices:

i: Index for component, $i \in \{1, \dots, N\}$

t: Index for time

j: Index for investment option, $j \in \{1..M_i\}$
k: Index for event option, $k \in \{1..K\}$

Variables:

$x_{i,j}$: A binary variable that is 1 if the investment option j, is selected for component q_i
$f_{i,j,t,k}$: Functionality of component i with investment $c_{i,j}$ at time t against γ_k
$f_{i,t}$: Functionality of component *i* at time *t*
B: Budget limit for enhancement against γ_k
A_{ik}: The drop in the functionality of the component i in face of event k

F_t: Functionality of system at time t, $F_t = F(t)$

Parameters:

$c_{i,j}$: The jth investment option of component i
RCI_i: Resilience-based component importance of q_i
T_i: Time to recovery of component *i* before investment
$T_{i,j,k}$: Time to recovery of component i after investment *j* against event *k*
Lf_i: Lowest functionality acceptable for a component
β_k: Weight of attack impact on resilience
$a_{i,j,k}$: Improvement in the absorption of component i in face of event k due to investment j.
$r_{i,j,k}$: Improvement in the recovery of component i in face of event k due to investment j.
β_{il}: Coefficients of the objective function $l = 1 ... 5$

Proper functionality of a system depends on the seamless functionality of its components. If the relationship between the functionality of a component and functionality of a system is given, then it may facilitate the measurement of the effect of component enhancement on the overall system resilience. However, it is very difficult to find such a straightforward relationship because the effect is a function of many unknown variables. To simplify the problem, it is assumed that functionality of the component at time *t*, $f_{i,t}$, has a *linear influence* on the functionality of the system, F_t, proportional to its resilience-based importance, RCI_i. That is

$$F(t) = F_t = \frac{1}{N}\sum_{i}^{N} RCI_i \times f_{i,t}. \tag{3}$$

Moreover, it is assumed that the component indifference curve for a given budget is known. The degradation time, t_d, is assumed to be the same for all of the components before and after the event. We start developing the mathematical model based on a piecewise linear function, and then move on to a more general model.

2.3.1. Linear functionality and linear utility curve for a single event

Assume that components have functionality (f_t) that consists of or can be estimated by two line segments l_1 and l_2 (Figure 3). To calculate the resilience using Equation (2), the integral is converted into the summation of the areas R_t captured by the trapezoid under the functionality curve in the interval $[t, t+1]$.

$$R_t = \int_t^{t+1} f(t)dt = \frac{f_{t+1} + f_t}{2}$$

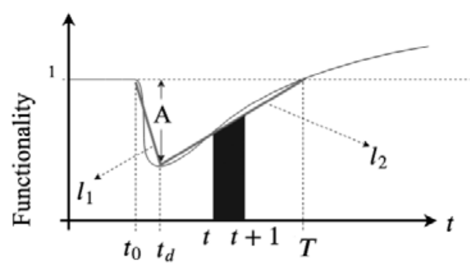

Figure 3: Estimation of the functionality of a component using two straight lines

At any time *t*, the functionality of a component is

$$f_{i,t} = \begin{cases} 1 - \dfrac{A_i}{t_{i,d}} t, & 0 \leq t \leq t_{i,d} \\ 1 - \dfrac{A_i}{T_i - t_d}(T_i - t), & t_{i,d} \leq t \leq T_i \\ 1, & t \geq T_i \end{cases}.$$

The sub-metrics absorption ($я_1$), adaptation ($я_2$), and recovery ($я_3$) sub-metrics can be calculated using $f_{i,t}$. In Appendix 1, we have derived formulas to calculate $я_1$ and $я_2$. If T_0 is small enough that $T \geq T_0$ is true, then the resilience of the system will be

$$я = \alpha_1 я_1 + \alpha_2 я_2 + \alpha_3 я_3$$

$$= \alpha_1 \frac{1}{N} \sum_{i=1}^{N} RCI_i(-\frac{1}{2}A_i + 1) + \alpha_2 \frac{1}{N} \sum_{i=1}^{N} RCI_i \left(\frac{-A_i}{2} \frac{(T_i - t_d)}{(T - t_d)} + 1 \right) + \alpha_3 \frac{T_0}{T}. \quad (4)$$

Equation (4) can be described visually using Figure 3. In this figure, the expression $-\frac{1}{2}A_i + 1$ is the area enclosed by the trapezoid with the corner points of $(0,0), (0,1), (t_d, 0), (t_d, 1 - A_i)$ over the area enclosed by the rectangle $(0,0), (0,1), (t_d, 0), (t_d, 1)$. The weighted average of these areas, where weights are RCI, yields $я_1$. A similar intuition holds for $\frac{-A_i}{2} \frac{(T_i - t_d)}{(T - t_d)} + 1$ and $я_2$. For a linear utility function and investment y_i, the new degradation and recovery time for component *i* are as follows:

$$A_{i,new} = A_i(1 - a), \text{and}$$

$$T_{i,new} = T_i(1 - r_i).$$

In Appendix 2, Equation (4) is modified to include the enhancements resulted from the investments as explained. Then we formulated the problem as a nonlinear optimization model described as in Equations (5-1) to (5-5).

$$\max \quad я = \frac{1}{N} \sum_{i=1}^{N} \left(\frac{\beta_{i1} a_i + \beta_{i2} a_i r_i + \beta_{i3} r_i + \beta_{i4} T + \beta_{i5} a_i T + \beta_{i6}}{T - t_d} \right) + \frac{T_0}{T} \quad (5\text{-}1)$$

s.t.

$$\sum_{i,j} y_i \leq B \quad (5\text{-}2)$$

$$\beta_{i1} a_i + \beta_{i2} r_i = k_i y_i, \forall i \quad (5\text{-}3)$$

$$T \geq T_i(1 - r_i), \forall i \quad (5\text{-}4)$$

$$y_i, T \geq 0, 0 \leq a_i, r_i \leq 1 \quad (5\text{-}5)$$

Decision variables are the budget allocated to component i (y_i), the percentage improvement in absorption and recovery time of the *component i* (a_i and r_i), and the *system*'s recovery time (T). Parameters include coefficients β_{ij}, which are calculated from parameters $\alpha_1, \alpha_2, RCI_i, A_i, T_i$, and t_d in Appendix 2; B the total system's budget limit; utility function parameters (γ_{i1}, γ_{i2}, and k_i); and time to recovery T_i for component *i* before an investment is made. The objective is to maximize the resilience of the system, Equation. (5-1), which is a function of component enhancements (a_i, r_i) and system's recovery time (T). Equation (5-2) restricts the total spending on the system to be less than the budget. Equation (5-3) relates the investment on component i (i.e., y_i) to different possibilities of enhancements in a_i and r_i via an indifference curve. As we discussed in Section 2.1, the scale factor k_i accounts for the value of the component. Recovery time of the system is the latest recovery time of the components (Equation (5-4)). Finally, all the variables are continuous and non-negative (Equation (5-5)), and a_i and r_i are less than 1. Solving this problem will determine how much we will spend on each component and what absorption and recovery enhancement combination will yield a higher resilience.

2.3.2. A General model

In this section, we extend the linearity assumption in Section 2.3.1 to general functionality and propose a mathematical programming model to optimize the system's resilience within a budget constraint. The goal is to allocate the budget to the components and to determine the component absorption and recovery enhancements in a such way that a maximum resilience is achieved. Drawing an analytical relationship between investment and system resilience is far reaching. To tackle this problem, we discretized the investment options and components' functionality through the following steps. In the first step, for each component q_i, m_i possible improvements $p_{i,j,k} = \left(a_{i,j,k}, r_{i,j,k}\right)$ and their associated cost $c_{i,j}$ is prepared. It is possible to have two different improvement scenarios $p_{i,j,k}$ and $p_{i,j\prime,k}$ associated with the same cost (i.e., $p_{i,j,k}$ and $p_{i,j\prime,k}$ lie on the same indifference curve). An improvement on q_i has different absorption and recovery enhancement outcomes against different events; shown by index k. All these data will be summarized in a set of options $O_{i,j,k} = (c_{i,j}, a_{i,j,k}, r_{i,j,k})$ among which the optimization problem chooses the subset of optimal options. For all the components, option $O_{i,0,k} = (0,0,0)$ is included so that it can be selected if no enhancement for component i is in optimal set. Using indifference curves, the construction of $O_{i,j,k}$ can be done either by finding different enhancement points $p_{i,j,k}$ for a given investment amount, or by finding the cost $c_{i,j}$ for a $p_{i,j,k}$.

In the second step, by considering $a_{i,j,k}$ and $r_{i,j,k}$, we obtain the discrete system functionality for each $O_{i,j,k}$ after a disruption. Let $f_{i,j,t,k}$ be the normalized functionality of q_i at time *t* after investment *j* in the face of the adversarial event $\gamma_k \in \Gamma$. The normalized functionality is calculated by dividing the actual functionality over the target functionality so that $f_{i,j,t,k}$ assumes a value between 0 and 1. Based on the linear influence assumption made at the beginning of Section 2.3, the influence of q_i on the total system's functionality is $CPI_i \times f_{i,j,t,k}$. Now, we use Equations (2) and (3) to construct the objective function in Equation (6-1), which is an indicator of resilience against the set of events. Equations (6-1) to (6-5) compose our general budget allocation model.

$$\max \quad \sum_{k} \beta_k \left[\begin{array}{c} \dfrac{\alpha_1 \sum_{t=0}^{t_{d,k}-1} \left(1 - \dfrac{1}{N} \sum_{i,j} CPI_i \times \dfrac{f_{i,j,t,k} + f_{i,j,t+1,k}}{2} \times x_{i,j} \right)}{t_{d,k} - t_0} + \\ \dfrac{\alpha_2 \sum_{t=t_d}^{T-1} \left(1 - \dfrac{1}{N} \sum_{i,j} CPI_i \times \dfrac{f_{i,j,t,k} + f_{i,j,t+1,k}}{2} \times x_{i,j} \right)}{T - t_{d,k}} + \\ \alpha_3 \dfrac{T_0}{T} \end{array} \right] \tag{6-1}$$

s.t.

$$\sum_{i,j} c_{i,j}\, x_{i,j} \leq B \tag{6-2}$$

$$\sum_{j} x_{i,j} = 1, \forall i \tag{6-3}$$

$$T \geq T_{i,j,k}\, x_{i,j}, \forall i \tag{6-4}$$

$$x_{i,j} \in \{0,1\} \tag{6-5}$$

The objective function (Equation (6-1)) is the weighted sum of the resilience of the system for the set of events Γ. The binary variable $x_{i,j}$ assumes 1 if the j^{th} investment option for q_i is selected. The weights β_k are parameters to show the importance of the corresponding event. They can be calculated using multi-criteria decision methods (MCDM) with criteria such as the possibility of event occurrence, cost of the devastation caused by the event, and cost of making the system resilient against that event. The budget constraint, Equation (6-2), limits the total cost of chosen options to be less than the budget. In the optimal solution, we just choose one investment option (Equation (6-3)) for each component. The key point in this equation is that we have designed the cost scenarios in such a way that we do not need to select two options for the same component for different events. Equation (6-4) calculates the system's time to recovery (T), which is the largest time to recovery of all the components. The steps for constructing the mathematical model is demonstrated in Algorithm 2.

Algorithm 2 Constructing the mathematical programming for general model

1. Input the set of components, simulation model to calculate functionality in different situations
2. For each component $i \in V \cup E$ do:
3. Determine the value k_i of the component from the V_i
4. Choose S values for the investments on component $i\,(c_i^s, s = 1, \ldots, S)$
5. Choose the utility function
6. $j = 0$
7. For each s in S:
    a. Determine P points on the indifference curve associated with C_i^s
    b. For each p in P:
        i. j =j+1
        ii. $c_{i,j} = c_i^s$
        iii. For k in events:
            a. Obtain $a_{i,j,k}$ and $r_{i,j,k}$ corresponding to point p on the IC of $c_{i,j}$ or find the cost associated with $(a_{i,j,k}, r_{i,j,k})$
            b. Calculate $f_{i,j,t,k}$ associated $a_{i,j,k}$ and $r_{i,j,k}$
8. Calculate the RCI_i using Algorithm 1
9. Insert these parameters into the model in Equations (6-1) to (6-5)

3. Numerical results

We perform our numerical studies on the power grid system using the security constrained unit commitment. We apply the SUSC model on a 6-bus IEEE standard test case. As shown in Figure 5, the grid comprises the generation units N0, N1, and N2; the electricity lines E3 to E9; and the demand nodes D4, D5, and D6 with share of total demand of 20%, 40%, and 40%, respectively. The 6-Bus data are available in Appendix 3.

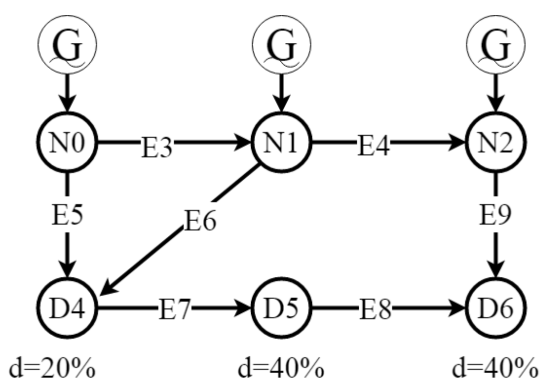

Figure 4: network of IEEE 6-Bus test system

By following the steps in Algorithm 2, we construct the data for our investment optimization problem for a single event ($k = 1$). For convenience, we drop the index x from our parameters (e.g., we use $O_{i,j}$ instead of $O_{i,j,k}$). For a component, say generator $N1$ ($i = 1$), we find the values of $O_{i,j}, j = 1, \dots, m_i$ and related functionalities $f_{i,j,t}$. The calculations for the rest of the components will be similar. We choose the scale factor k_i to be the ratio $1/V_i$, which indicates that the investment will be proportionate to the value of the component. If we assume that the value of a brand new generator $N1$ is \$30,000 ($V_1 = 30000$), the value of k_1 will be 1/30000. If we invest \$7,500 on this generator (i.e., $y_1 = 7500$), then the cost factor θ_1 will be 0.25 ($\theta_1 = k_1 y_1 = 0.25$). In case both θ_i and V_i are given, then we can simply calculate the investment cost by $y_i = \theta_i V_i$. Associated with \$7,500 investment and a linear utility curve with $\gamma_{1,1}$ and $\gamma_{1,2}$ both equal to 0.5, the indifference curve in Figure 5 is constructed. Now we can choose as many points $p_{1,j} = \left(a_{1,j}, r_{1,j}\right)$ on this curve as we need (e.g., $p_{1,1} = (0.5{,}0), p_{1,2} = (0{,}0.5)$, and $p_{1,3} = (0.25{,}0.25)$ with corresponding options of $O_{1,1} = (7500, 0.5{,}0), O_{1,2} = (7500{,}0{,}0.5)$, and $O_{1,3} = (7500{,}0.25{,}0.25)$). For a higher investment amount, say \$22,500, the cost factor θ_1 is 0.75 which provides better options such as $O_{1,4} = (22500{,}1.0{,}0.5)$ and $O_{1,5} = (22500{,}0.5{,}1.0)$. The above process constructs $p_{i,j}$ for a given $c_{i,j}$ (another way is to find the $c_{i,j}$ associate for a $p_{i,j}$ by first obtaining the θ_i using the utility function, and then multiplying it by V_i.) No matter whether we construct costs from enhancements or enhancements from cost, the goal in this step is to construct the investment options $O_{i,j}$, among which the decision maker will choose.

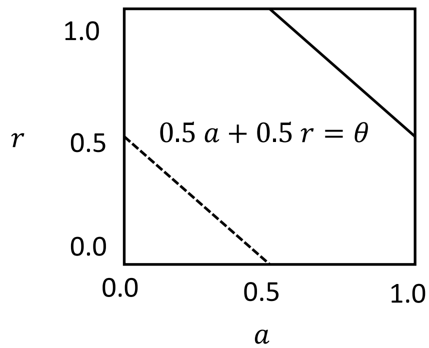

Figure 5: change in the absorption and recovery of component i against event k for a) cost factor of 0.25 and b) cost factor of 1.5

Following the second way, we chose $p_{i,j}$ as in Figure 6. For each component, the scenario $p_{i,0} = (0,0)$ is added and it stands for no enhancement option. The associated cost of these enhancement for a linear utility with γ_{i1} and γ_{i2} of 0.5 is summarized in the heat-map chart in Figure 7. In this figure darker cells indicates a higher investment (e.g., the cost of enhancing the component N1 by $(\mathrm{a}, \mathrm{r}) = (1,0.75)$ is 60). For each option $O_{i,j}$, we obtain the resulted functionality $f_{i,j,t}$ by considering the corresponding absorption and recovery.

r \ a	0	0.25	0.5	0.75	1
1		(0.25,1)	(0.5,1)	(0.75,1)	(1,1)
0.75		(0.25,0.75)	(0.5,0.75)	(0.75,0.75)	(1,0.75)
0.5		(0.25,0.50)	(0.5,0.5)	(0.75,0.5)	(1,0.5)
0.25		(0.25,0.25)	(0.5,0.25)	(0.75,0.25)	(1,0.25)
0	(0,0)				

Figure 6: $p_{i,j}$ scenarios

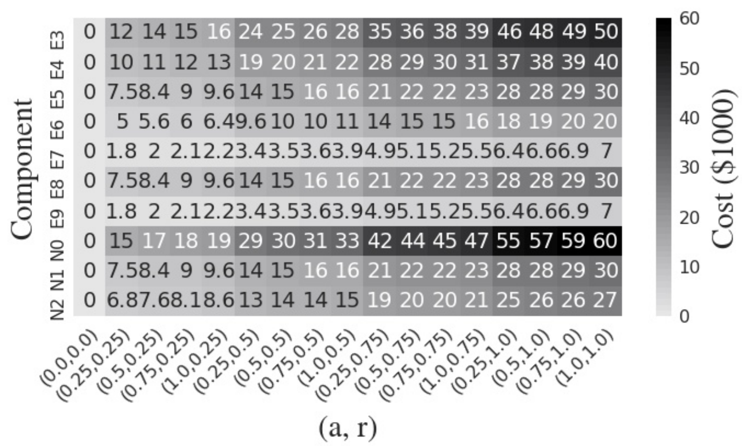

Figure 7: heat-map of component investment options for a linear utility function with parameters γ_{i1} and γ_{i2} of 0.5 and 0.5 respectively. N1, N2, and N3 are nodes and E3 to E10 are electricity lines between the nodes. The color bar in the right shows the value cells in the heat map.

Utilizing the SCUC and the resilience metric in [35], Algorithm 1 outputs the RCI as in Figure 8. A failure in the component E9 has a higher impact on the resilience of the system and the component N2 disruption has the lowest effect. Now, the parameters $c_{i,j}, f_{i,j,t}, RCI_i,$ and $t_{i,j}$ for the problem Equations (6-1) to (6-5) are ready. Applying these parameters and optimizing the budget allocation for a budget limit of \$50,000 yields the solution in Table 2. We calculate the resilience metric for the optimal $P_{i,j}$. Let the initial resilience level of the system be $я = 0.73$, which happens when the absorption of all the components drop to 50% after an event and the recovery time is the same as mean time to repair of the component. The optimal investment in Table 2 with a total cost of \$49,500 improves the resilience level to $я = 0.84$.

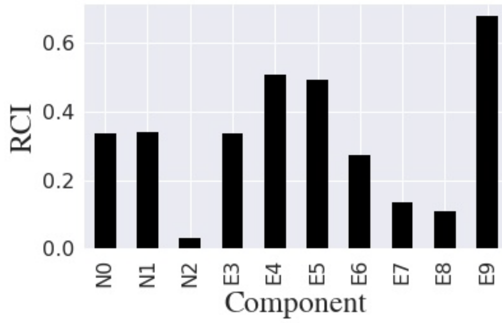

Figure 8: component importance for IEEE 6-bus

Table 2 Optimal investments for linear utility with both parameters of 0.5 and budget limits of $50,000. The resulting total cast is $49,500 and the resulted objective is 0.372.

	N0	N1	N2	E3	E4	E5	E6	E7	E8	E9
$c^*_{i,j}$ **($1000)**	0	11.4	0	12.5	0	11.4	7.6	2.7	0	4.3
$a^*_{i,j,k}$	0	0.25	0	0.25	0	0.25	0.25	0.25	0	1
$r^*_{i,j,k}$	0	0.5	0	0.25	0	0. 5	0.5	0. 5	0	0.25

To find out the effect of different budget limits, further experiments are made on different budget scenarios. Considering that the value of the existing system is $301,000 ($\sum_{i=1}^{N} V_i$), we chose the values of $10,000, $50,000, $100,000, $150,000, and 300,000 for the budget limit. Following the steps for all these budgets, we will find the corresponding improvement in the resilience of the system (Figure 9). For the budget of $150,000 and $300,000 the optimal solution yields the same amount of investment ($103,800) and the optimal solution remains the same. Hence, the highest budget that is needed to enhance the resilience of the 6-bus system with linear utility whose parameters are $(\gamma_{i1}, \gamma_{i2}) = (0.5,0.5)$ is $103,800.

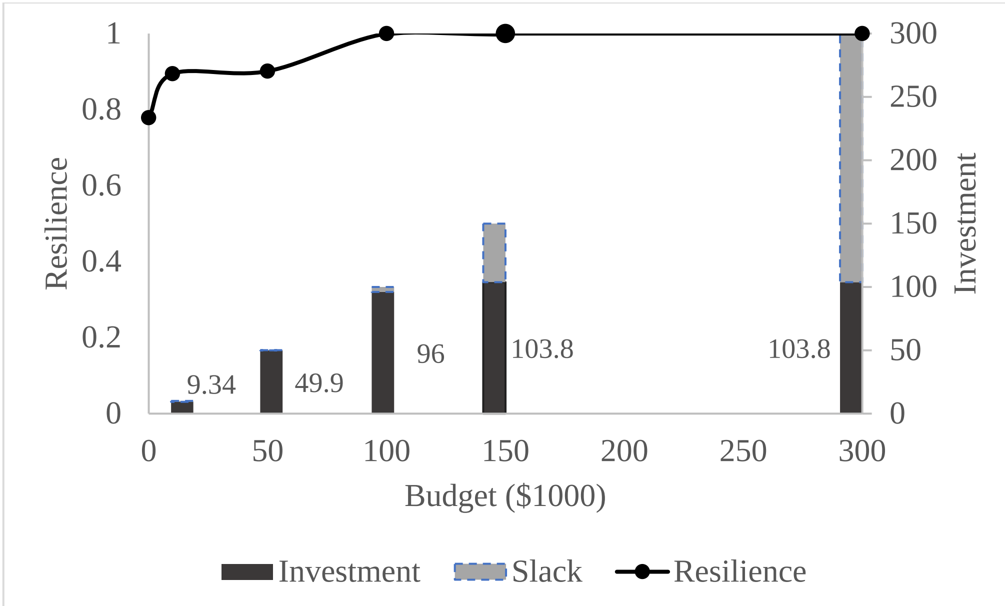

Figure 9 Resilience improvement in for budget limits of $10,000, $50,000, $100,000, $150,000, and 300,000 and a linear utility function with $(\gamma_{1,1}, \gamma_{1,2}) = (0.5, 0.5)$.

The utility function and its parameters are inherent characteristics of a component. Knowing a utility function can help us to decide about the type of improvement in the component. Table 3 summarizes the $\theta_{i,j}$ for utility functions introduced in Table 1 with different parameters and $p_{i,j}$. Assuming that all the components have the same utility function, Algorithm 2 is applied for each utility function and budget limits of $10,000, $50,000, $100,000, $150,000, and 300,000. After obtaining the optimal allocation, the resulted resiliency measure is obtained as in Figure 11-13. The linear utility functions with parameters (γ_1, γ_2) of $(0.7, 0.3)$ and $(0.5, 0.5)$ show a higher cost and a lower resilience for budgets of $10,000 and $50,000. However, the amount of investment in the components can change this behavior. Figure 13 shows the changes in the area under the functionality curves for different coefficients and cost factor θ. As θ increases the utility function shifts to the upper right (left-hand side plots in Figure 14), meaning that better absorption and recovery combinations are possible. The area under the functionality curve, as a sub-metric of the resilience metric, shows highest value for the $a = 1$. For an investment of 90% of the total value of the component (i.e., $\theta = 0.9$), the area under the utility curve approaches to 1, which means that the component is close to its highest functionality after an event. Based on Figure 11, the cost of enhancing resilience of a system whose components have a linear utility function with parameters of (0.9,0.1) is lower.

The Cobb-Douglas utility (Figure 13) with $\rho = 0.3$ and $\rho = 0.5$ resulted in similar resilience and investment costs. A $\rho = 0.1$ has a resilience similar to $\rho = 0.3$ and $\rho = 0.5$ but with lower costs for budget limits over $\$100{,}000$. The other two parameters ($\rho = 0.7$ and $\rho = 0.9$) resulted in a lower cost and a higher resilience. If components have a CES utility with parameters (γ, ρ) of (0.5,1), then system will reach to its highest resilience level for a budget greater than $\$100{,}000$, while this budget limit is $\$50{,}000$ for other parameters. If the goal is to increase the resilience of the system from $я = 0.73$ to a resilience around $я = 0.9$, then CES with parameters (0.3, 0.1) would need a more budget.

Table 3 Cost factor $\theta_{i,j}$ of absorption and recovery scenarios for utility functions

		$\boldsymbol{p_{i,j,k}} = (a_{i,j,k}, r_{i,j,k})$																
Utility function	**parameters**	**(0,0)**	**(0.25,0.25)**	**(0.25,0.5)**	**(0.25,0.75)**	**(0.25,1)**	**(0.5,0.25)**	**(0.5,0.5)**	**(0.5,0.75)**	**(0.5,1)**	**(0.75,0.25)**	**(0.75,0.5)**	**(0.75,0.75)**	**(0.75,1)**	**(1,0.25)**	**(1,0.5)**	**(1,0.75)**	**(1,1)**
Linear (γ_1, γ_2)	**(0.1, 0.9)**	0	0.25	0.48	0.7	0.92	0.28	0.5	0.73	0.95	0.3	0.52	0.75	0.98	0.32	0.55	0.78	1
	(0.3, 0.7)	0	0.25	0.42	0.6	0.77	0.32	0.5	0.68	0.85	0.4	0.57	0.75	0.92	0.48	0.65	0.82	1
	(0.5, 0.5)	0	0.25	0.38	0.5	0.62	0.38	0.5	0.62	0.75	0.5	0.62	0.75	0.88	0.62	0.75	0.88	1
	(0.7, 0.3)	0	0.25	0.32	0.4	0.48	0.42	0.5	0.57	0.65	0.6	0.68	0.75	0.82	0.77	0.85	0.92	1
	(0.9, 0.1)	0	0.25	0.28	0.3	0.32	0.48	0.5	0.52	0.55	0.7	0.73	0.75	0.78	0.92	0.95	0.98	1
Cobb-Douglas ρ	**0.1**	0	0.25	0.47	0.67	0.87	0.27	0.5	0.72	0.93	0.28	0.52	0.75	0.97	0.29	0.54	0.77	1
	0.3	0	0.25	0.41	0.54	0.66	0.31	0.5	0.66	0.81	0.35	0.56	0.75	0.92	0.38	0.62	0.82	1
	0.5	0	0.25	0.35	0.43	0.5	0.35	0.5	0.61	0.71	0.43	0.61	0.75	0.87	0.5	0.71	0.87	1
	0.7	0	0.25	0.31	0.35	0.38	0.41	0.5	0.56	0.62	0.54	0.66	0.75	0.82	0.66	0.81	0.92	1
	0.9	0	0.25	0.27	0.28	0.29	0.47	0.5	0.52	0.54	0.67	0.72	0.75	0.77	0.87	0.93	0.97	1
CES (β, ρ)	**(0.1, 0.5)**	0	0.25	0.47	0.69	0.9	0.27	0.5	0.72	0.94	0.29	0.52	0.75	0.97	0.3	0.54	0.77	1
	(0.3, 0.1)	0	0.25	0.41	0.55	0.67	0.31	0.5	0.67	0.82	0.35	0.57	0.75	0.92	0.39	0.62	0.82	1
	(0.5, 1)	0	0.25	0.38	0.5	0.62	0.38	0.5	0.62	0.75	0.5	0.62	0.75	0.88	0.62	0.75	0.88	1
	(0.7, 0.3)	0	0.25	0.31	0.36	0.4	0.41	0.5	0.57	0.63	0.56	0.67	0.75	0.82	0.7	0.82	0.92	1
	(0.9, 0.4)	0	0.25	0.27	0.29	0.3	0.47	0.5	0.52	0.54	0.69	0.72	0.75	0.77	0.9	0.94	0.97	1

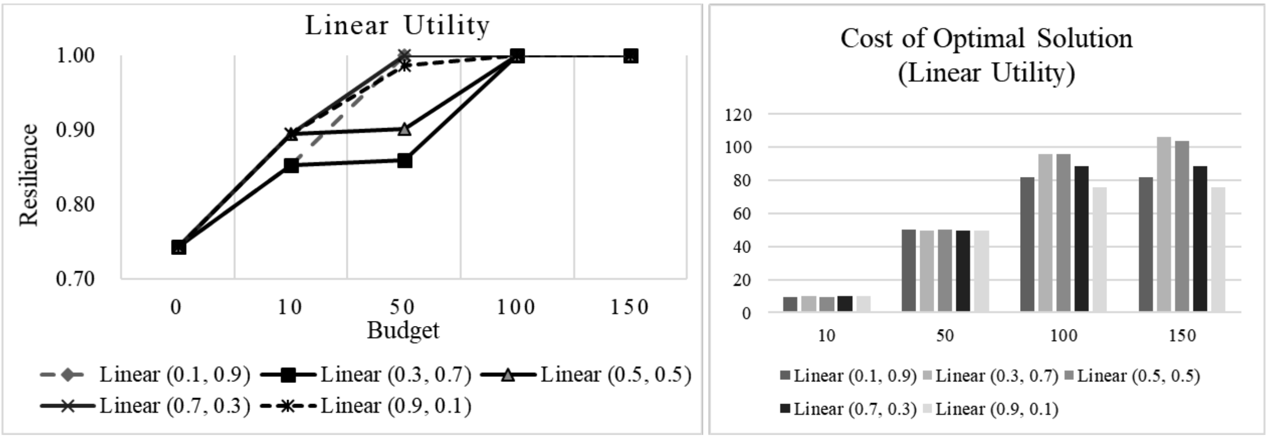

Figure 10 Resilience achieved by applying the enhancement in the optimal solution for different budgets for linear utility *functions.*

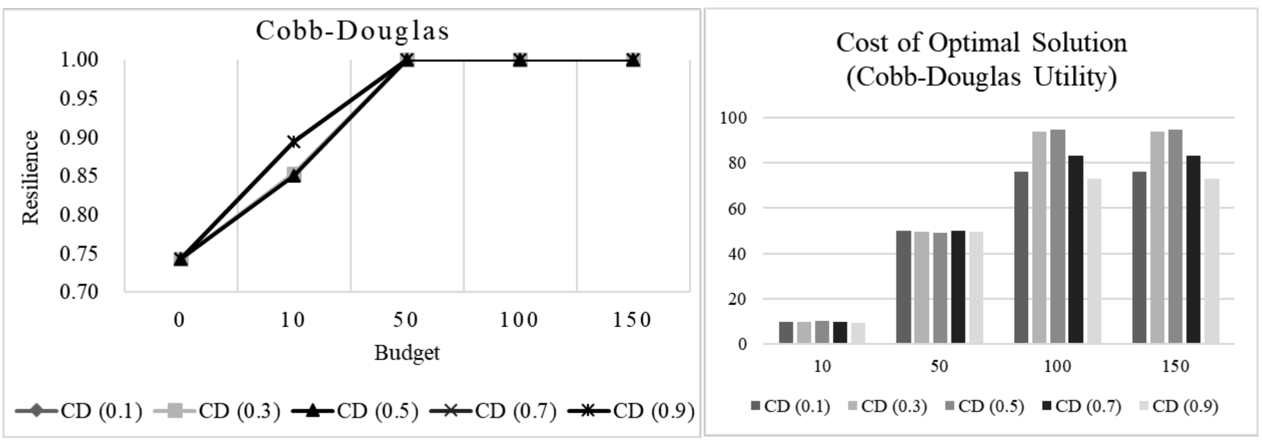

Figure 11 Resilience achieved by applying the enhancement in the optimal solution for different budgets for Cobb-Douglas utility *functions.*

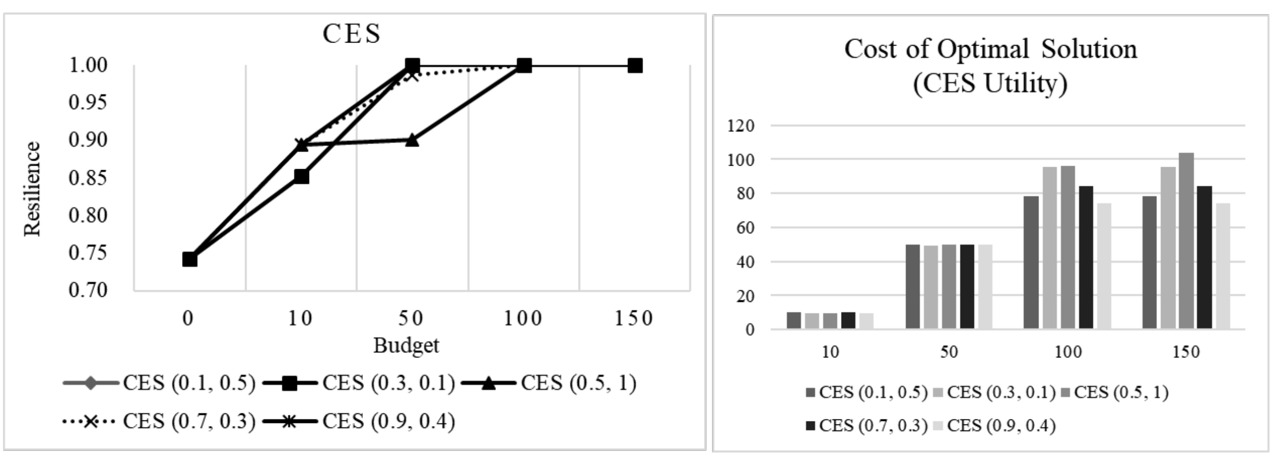

Figure 12 Resilience achieved by applying the enhancement in the optimal solution for different budgets for CES utility functions.

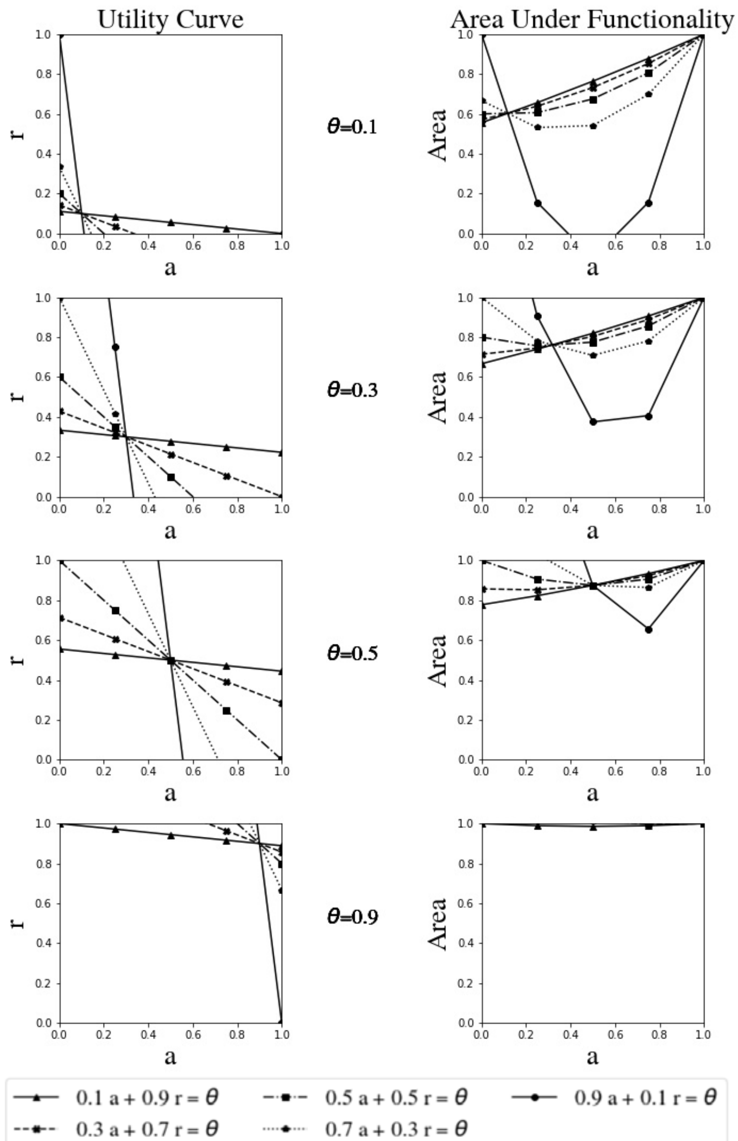

Figure 13 charts on the left are linear utility curves for different values of investment and parameters. Charts on the right are the area under utility function for the utility functions on the left.

In summary, if components of the system we considered in our numerical analysis, all have a Cobb-Douglas utility curve with $\rho = 0.9$, then the system will become more resilient with a lower investment comparing to other utility functions. If we have two components with the same functionality and cost, but with two different utility curves, we can use similar analysis to choose one them. For a budget limit and known utility functions, Algorithm 2 can give the optimal budget allocation to maximize the resilience.

4. Conclusion

In this study, we considered the infrastructure investment under budget constraint. For this purpose the resilience metric suggested by [35] is used. Different states of enhancements for component's absorption and recovery were elaborated and a metric was introduced to measure the resilience-based component importance. The mathematical programming formulation was suggested to allocate budget to components while maximizing the resilience within that budget. In this process, we introduced a novel method to bypass the complexity of the system by discretizing the options. The introduced optimization problem determines how much to be invested in each component, and what are the optimal level of enhancement in component's absorption and recovery time. We applied our budget allocation problem on the IEEE 57-bus and discussed the effect of different utility curves on the cost of enhancement and system's resilience level. The results show that for bus-6 test case, we need less than half the value of the system to improve its resilience from 0.73 to 1. Moreover, the resilience of a system whose components have a Cobb-Douglas utility function may be enhanced with a lower budget. The relationship between the change in the component absorption and recovery and the change in the system functionality is assumed to be linear. Future studies will concentrate on finding a system specific relationship between different enhancements scenarios and the change in the functionality of the system in the face of adverse events. It will help to find a more accurate evaluation of the system resilience after investment.

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Appendix 1: Calculations of $я_1$ and $я_2$ before investment

(a) Calculations for $я_1$

$$я_1 = \int_0^{t_d} \frac{F(t)}{TF(t)} dt = \sum_{t=0}^{t_d-1} \frac{F(t)+F(t+1)}{2} = \frac{1}{2t_d} \sum_{t=0}^{t_d-1} \left(\frac{1}{N} \sum_{i=1}^{N} RCI_i \, f_{i,t} + \frac{1}{N} \sum_{1}^{N} RCI_i \, f_{i,t+1} \right)$$

$$= \frac{1}{2Nt_d} \sum_{t=0}^{t_d-1} \left(\sum_{i=1}^{N} RCI_i \left(\left(1 - \frac{A_i}{t_d} t \right) + \left(1 - \frac{A_i}{t_d} (t+1) \right) \right) \right)$$

$$= \frac{1}{2Nt_d} \sum_{i=1}^{N} \left(RCI_i \sum_{t=0}^{t_d-1} \left(2 - \frac{2A_i}{t_d} t - \frac{A_i}{t_d} \right) \right) = \frac{1}{Nt_d} \sum_{i=1}^{N} RCI_i \left(-\frac{1}{2} A_i t_d + t_d \right)$$

$$= \frac{1}{N} \sum_{i=1}^{N} RCI_i \left(-\frac{1}{2} A_i + 1 \right)$$

(b) Calculations for $я_2$

$$я_2 = \int_{t_d}^{T} \frac{F(t)}{TF(t)} dt = \frac{1}{T - t_d} \sum_{t=t_d}^{T-1} \frac{F(t)+F(t+1)}{2}$$

$$= \frac{1}{2(T-t_d)} \sum_{t=t_d}^{T-1} \left(\frac{1}{N} \sum_{i=1}^{N} RCI_i \, f_{i,t} + \frac{1}{N} \sum_{i=1}^{N} RCI_i \, f_{i,t+1} \right)$$

$$= \frac{1}{2N(T-t_d)} \sum_{i=1}^{N} RCI_i \left(\sum_{t=t_d}^{T-1} f_{i,t} + \sum_{t=t_d+1}^{T} f_{i,t+1} \right)$$

We calculate $\sum_{t=t_d}^{T-1} f_{i,t} + \sum_{t=t_d+1}^{T} f_{i,t+1}$ separately and plug it into the above formula.

$$\sum_{t=t_d}^{T-1} f_{i,t} = \sum_{t=t_d}^{T_i-1}\left(1 - \frac{-A_i}{T_i - t_d}(T_i - t)\right) + \sum_{t=T_i}^{T-1} 1$$

$$= \sum_{t=t_d}^{T_i-1} \frac{-A_i}{T_i - t_d} T_i + \sum_{t=t_d}^{T_i-1} \frac{A_i}{T_i - t_d} t + \sum_{t=t_d}^{T-1} 1$$

$$= -A_i T_i + \frac{A_i}{2}(t_d + T_i - 1) + (T - t_d),$$

and

$$\sum_{t=t_d+1}^{T} f_{i,t+1} = \sum_{t=t_d+1}^{T_i}\left(1 - \frac{A_i}{T_i - t_d}(T_i - t)\right) + \sum_{t=T_i+1}^{T} 1$$

$$= \sum_{t=t_d+1}^{T_i} \frac{-A_i}{T_i - t_d} T_i + \sum_{t=t_d+1}^{T_i} \frac{A_i}{T_i - t_d} t + \sum_{t=t_d+1}^{T} 1$$

$$= -A_i T_i + \frac{A_i}{2}(t_d + T_i + 1) + (T - t_d).$$

Hence

$$\sum_{t=t_d}^{T-1} f_{i,t} + \sum_{t=t_d+1}^{T} f_{i,t+1}$$

$$= -A_i T_i + \frac{A_i}{2}(t_d + T_i - 1) + (T - t_d) - A_i T_i + \frac{A_i}{2}(t_d + T_i + 1) + (T - t_d)$$

$$= -2A_i T_i + \frac{A_i}{2}(2t_d + 2T_i) + 2(T - t_d) = -A_i(T_i - t_d) + 2(T - t_d)$$

Replacing the $\sum_{t=t_d}^{T-1} f_{i,t} + \sum_{t=t_d+1}^{T} f_{i,t+1}$ in the $я_2$ yields

$$я_2 = \frac{1}{2N(T - t_d)} \sum_{i=1}^{N} RCI_i\big(-A_i(T_i - t_d) + 2(T - t_d)\big) = \frac{1}{N}\sum_{i=1}^{N} RCI_i\left(\frac{-A_i}{2}\frac{(T_i - t_d)}{(T - t_d)} + 1\right).$$

Appendix 2: Calculating я after investment

The parameters $\alpha_1, \alpha_2, RCI_i, A_i, T_i, \gamma_{i,2}$, and t_d are inputs to the model and they are known prior to the optimization. Let $b_{i1} = \frac{\alpha_1 RCI_i A_i}{2}$, $b'_{i1} = -b_{i1} + \alpha_1 RCI_i t_d$, $b_{i2} = \frac{\alpha_2 RCI_i A_i}{2}$, and $b'_{i2} = b_{i2} T_i$, then $я_1$ and $я_2$ can be calculated as follows:

$$я_1 = \frac{1}{N} \sum_{i=1}^{N} RCI_i \left(-\frac{1}{2} A_i (1 - a_i) + 1 \right) = \frac{1}{N} \sum_{1}^{N} (b_{i1} a_i + b'_{i1})$$

$$я_2 = \frac{1}{N} \sum_{i=1}^{N} RCI_i \left(\frac{-A_i (1 - a_i)}{2} \frac{[T_i\ (1 - r_i) - t_d]}{(T - t_d)} + 1 \right)$$

$$= \frac{1}{N} \sum_{i=1}^{N} \frac{-\alpha_2 RCI_i A_i}{2} \frac{[T_i\ (1 - r_i) - t_d]}{(T - t_d)}$$

$$+ \frac{1}{N} \sum_{i=1}^{N} \frac{\alpha_2 RCI_i A_i}{2} \frac{a_i [T_i\ (1 - r_i) - t_d]}{(T - t_d)} + \alpha_2 RCI_i$$

$$= \frac{1}{N} \sum_{i=1}^{N} \left(\frac{[b'_{i2}\, r_i - b'_{i2} + b_{i2} t_d] + [\, b'_{i2}\, a_i - b'_{i2}\, r_i a_i + b_{i2} t_d a_i]}{T - t_d} \right) + \alpha_2 RCI_i$$

$$= \frac{1}{N} \sum_{i=1}^{N} \left(\frac{b'_{i2}\, r_i + (b'_{i2} + b_{i2} t_d) a_i - b'_{i2}\, r_i a_i + (b_{i2} t_d - b'_{i2})}{T - t_d} \right) + \alpha_2 RCI_i$$

$$я = \alpha_1 я_1 + \alpha_2 я_2 + \alpha_3 \frac{T_0}{T} = \frac{1}{N} \sum_{i=1}^{N} \left(\frac{\beta_{i1}\, a_i + \beta_{i2}\, a_i r_i + \beta_{i3}\, r_i + \beta_{i4} T + \beta_{i5} a_i T + \beta_{i6}}{T - t_d} \right) + \frac{T_0}{T}$$

where $\beta_{i1} = b_{i1} t_d + b'_{i2} + b_{i2} t_d$, $\beta_{i2} = -b'_{i2}$, $\beta_{i3} = b'_{i2}$, $\beta_{i4} = b'_{i1}$, $\beta_{i5} = b_{i1}$ and $\beta_{i6} = b'_{i1} t_d + b_{i2} t_d - b'_{i2}$.

Appendix 3: IEEE 6-Bus data

Figure A1 shows the Bus-6 network configuration, and Figure A2 shows the hourly power demand for this test network. Table 1 contains power generation data, and Table 2 contains data for the transmission lines.

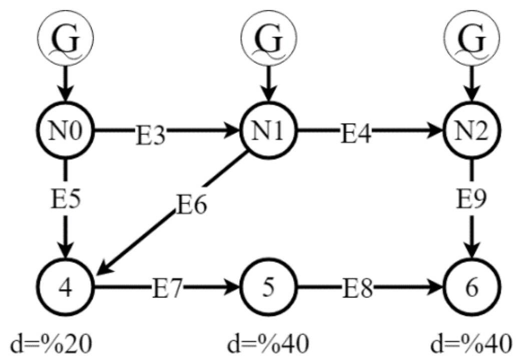

Figure A1: Bus-6 network diagram

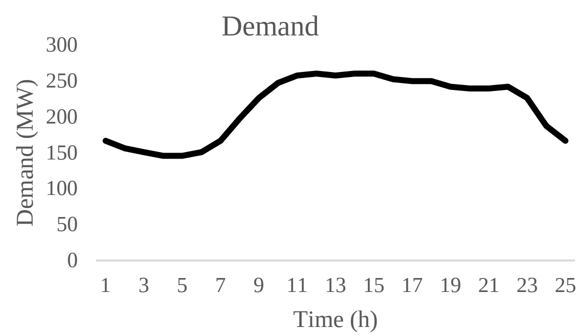

Figure A2: Bus-6 hourly total demand

Table A1: Bus-6 data for generation units.

Name	bus	pmin	pmax	cnl	sdc	suc	mu	md	ru	rd	msr	qsc	laststat	mn	lastp	mf	psegmax1	psegmax2	cseg1	cseg2	genmttr	investment
N0	1	100	220	10.15	50	100	4	4	55	55	1	0	1	5	25	25	50	50	18.46	55.4	30	60
N1	2	10	100	39	100	200	3	2	50	50	1	0	1	11	50	25	100	100	13.73	41.2	68	30
N2	3	10	40	31.67	0	0	1	1	20	20	1	0	1	1	10	25	15	15	27.2888	81.9	68	27

Table A2: Bus-6 data for transmission lines.

Names	begin	end	phase	capacity	linemttr	investment
E3	1	2	0.17	200	12	10
E4	2	3	0.037	100	8	7
E5	1	4	0.258	100	8	7
E6	2	4	0.197	100	8	7
E7	4	5	0.037	100	8	7
E8	5	6	0.14	100	8	7
E9	3	6	0.018	100	8	7

Appendix 4: Cost Factors

	re	**0**	**0.25**				**0.5**				**0.75**				**1**			
	ab	**0**	**0.25**	**0.5**	**0.75**	**1**	**0.25**	**0.5**	**0.75**	**1**	**0.25**	**0.5**	**0.75**	**1**	**0.25**	**0.5**	**0.75**	**1**
	(0.75,0.25)	0	0.25	0.31	0.38	0.44	0.44	0.5	0.56	0.62	0.62	0.69	0.75	0.81	0.81	0.88	0.94	1
	(0.6,0.4)	0	0.25	0.35	0.45	0.55	0.4	0.5	0.6	0.7	0.55	0.65	0.75	0.85	0.7	0.8	0.9	1
	(0.5,0.5)	0	0.25	0.38	0.5	0.62	0.38	0.5	0.62	0.75	0.5	0.62	0.75	0.88	0.62	0.75	0.88	1
	(0.4,0.6)	0	0.25	0.39	0.54	0.68	0.36	0.5	0.64	0.79	0.46	0.61	0.75	0.89	0.57	0.71	0.86	1
Linear	**(0.25,0.75)**	0	0.25	0.41	0.56	0.72	0.34	0.5	0.66	0.81	0.44	0.59	0.75	0.91	0.53	0.69	0.84	1
	0.1	0	0.25	0.27	0.28	0.29	0.47	0.5	0.52	0.54	0.67	0.72	0.75	0.77	0.87	0.93	0.97	1
	0.3	0	0.25	0.31	0.35	0.38	0.41	0.5	0.56	0.62	0.54	0.66	0.75	0.82	0.66	0.81	0.92	1
	0.5	0	0.25	0.35	0.43	0.5	0.35	0.5	0.61	0.71	0.43	0.61	0.75	0.87	0.5	0.71	0.87	1
	0.9	0	0.25	0.47	0.67	0.87	0.27	0.5	0.72	0.93	0.28	0.52	0.75	0.97	0.29	0.54	0.77	1
	(0.1, 0.1)	0	0.25	0.27	0.28	0.29	0.47	0.5	0.52	0.54	0.68	0.72	0.75	0.77	0.88	0.94	0.97	1
	(0.3, 0.5)	0	0.25	0.36	0.45	0.54	0.36	0.5	0.62	0.72	0.45	0.62	0.75	0.87	0.54	0.72	0.87	1
	(0.5, 0.4)	0	0.25	0.34	0.42	0.49	0.39	0.5	0.59	0.68	0.52	0.64	0.75	0.85	0.64	0.78	0.9	1
CES	**(0.9, 1)**	0	0.25	0.5	0.75	1	0.25	0.5	0.75	1	0.25	0.5	0.75	1	0.25	0.5	0.75	1